Recurrent graphs where two independent random walks collide finitely often

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Abstract

We present a class of graphs where simple random walk is recurrent, yet two independent walkers meet only finitely many times almost surely. In particular, the comb lattice, obtained from $\mathbf{Z}^2$ by removing all horizontal edges off the x-axis, has this property. We also conjecture that the same property holds for some other graphs, including the incipient infinite cluster for critical percolation in $\mathbf{Z}^2$.

1 Introduction

In "Two Incidents" [7], George Pólya describes the incident that led him to his celebrated results on random walks on Euclidean lattices:

> *"... he and his fiancée (would) also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly, I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case. I met them by accident - but how likely was it that it happened by accident and not on purpose?"*

Pólya formulated the problem of the meeting of two walkers for random walks on a Euclidean lattice; in that case, it reduces to the problem of a single walker returning to his starting point. As we show in this paper, these two problems can have different answers when the ambient graph is not transitive.

Call a graph $\mathbf{G}$ recurrent if simple random walk on it is recurrent. Say that a graph $\mathbf{G}$ has the **finite collision property** if two independent simple random walks X, Y on $\mathbf{G}$ starting from the same vertex meet only finitely many times, i.e., $|\{n : X_n = Y_n\}| < \infty$, almost surely. Our goal is to present a class of recurrent graphs with the finite collision property.

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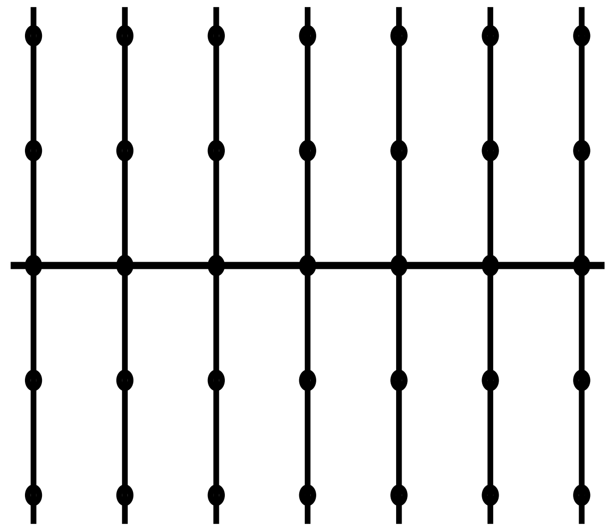

Figure 1: Comb lattice (Comb($\mathbf{Z}$))

Definition Given a graph $\mathbf{G}$, let Comb($\mathbf{G}$) be the graph with vertex set $V(\mathbf{G}) \times \mathbf{Z}$ and edge set

$$\{[(x,n),(x,m)] : |m-n| = 1\} \cup \{[(x,0),(y,0)] : [x,y] \text{ is an edge in } \mathbf{G}\}.$$

In words, this means that we attach a copy of $\mathbf{Z}$ at each vertex of the graph $\mathbf{G}$. See Fig 1 for a picture of Comb($\mathbf{Z}$). Clearly, if $\mathbf{G}$ is recurrent, so is Comb($\mathbf{G}$).

Theorem 1.1 *Let $\mathbf{G}$ be any recurrent infinite graph with constant vertex degree. Then Comb($\mathbf{G}$) has the finite collision property.*

We note the following points.

- Liggett [5] has given examples of symmetric recurrent Markov chains for which two independent copies of the chain collide only finitely many times. Those examples are not simple random walks on graphs, however.

- If X, Y are independent random walks on a graph starting from a vertex v, then the expected number of meetings between them is $\sum_n \sum_w (p^{(n)}(v,w))^2$, where $p^{(n)}$ is the n-step transition function. Now,

$$p^{(2n)}(v,v) = \sum_w (p^{(n)}(v,w))^2 \frac{\pi(v)}{\pi(w)},$$

  where $\pi(w)$ denotes the degree of w. Therefore, if the expected number of meetings between two independent walkers is finite, so is $\sum_n p^{(2n)}(v,v)$, and hence the graph is transient.

  The converse is true for bounded degree graphs because then $\frac{\pi(v)}{\pi(w)}$ is bounded away from zero. This converse can fail when the degrees are unbounded as the following example shows: Consider $\mathbf{Z}_+$ and add 2^n disjoint paths of length 2 between n and $n+1$. Then, X_{2n} and Y_{2n} are just random walks on $\mathbf{Z}_+$ with a bias of $\frac{1}{3}$ to the right and hence the graph is transient. However they meet infinitely often

almost surely (the difference eventually coincides with an unbiased random walk on $\mathbf{Z}$ and hence visits zero infinitely often).

- Recurrent transitive graphs cannot have the finite collision property. This is because transitivity clearly implies that the number of meetings has a Geometric distribution, whence it is finite only if it has finite expectation.

2 Proof of Theorem 1.1

Proof of Theorem 1.1 As noted earlier, Comb($\mathbf{G}$) is recurrent. We only need to prove the finite collision property. Let X and Y be independent simple random walks (SRWs) on Comb($\mathbf{G}$) starting from the same vertex $(o, 0)$. We make the following definitions.

$$Z_{n,\ell} := |\{(N, L) : n \le N \le 2n, \ell \le L \le 2\ell \text{ and } X_N = Y_N = (v, L) \text{ for some } v \in \mathbf{G}\}|.$$
$$A_{n,\ell} := \{Z_{n,\ell} > 0\}.$$
$$W_{n,\ell} := \sum_{k=\frac{\ell}{2},\ell,2\ell} Z_{n,k} + \sum_{k=\frac{\ell}{2},\ell,2\ell} Z_{2n,k}.$$

(Here $|S|$ denotes the number of elements of S.)

Let d denote the common degree of vertices in $\mathbf{G}$. In what follows, C, C_1, C_2 etc. will denote positive finite constants whose values may change from one appearance to another.

Lemma 2.1 $\mathbf{E}[Z_{n,\ell}] \le C\ell n^{-1/4}$ *for some finite constant* C, $\forall n, \ell \ge 1$.

Remark For the case when $\mathbf{G} = \mathbf{Z}$, i.e., for Comb($\mathbf{Z}$), this lemma is suggested by the fact that

$$p^{(2n)}(0,0) \sim \frac{\sqrt{2}}{\Gamma(1/4)} n^{-3/4}. \tag{1}$$

See Proposition 18.4 in Woess [8] for a proof of (1).

Proof (Lemma 2.1) We generate the random walks X and Y in the following manner. Let U and U' be independent simple random walks on $\mathbf{G}$ starting from a vertex o. Let V and V' be independent simple random walks on $\mathbf{Z}$, with the modification that they have a self-loop probability of $\frac{d}{d+2}$ at 0. Let K_n and K'_n be the number of transitions of V and V' from 0 to 0 in the first n steps. Then set

$$X_n = (U_{K_n}, V_n) \text{ and } Y_n = (U'_{K'_n}, V'_n).$$

It is clear that X and Y are independent simple random walks on Comb($\mathbf{G}$), both starting from $(o, 0)$.

Now fix any $L \in \mathbf{Z}$ and consider

$$\begin{aligned}\mathbf{P}[X_n = Y_n = (v, L) \text{ for some } v \in \mathbf{G}] &= \sum_{k=0}^{\infty}\sum_{k'=0}^{\infty} \mathbf{P}[V_n = V'_n = L; K_n = k, K'_n = k'; U_k = U'_{k'}] \\ &= \sum_{k,k'} \mathbf{P}[V_n = V'_n = L; K_n = k, K'_n = k']\mathbf{P}[U_k = U'_{k'}].\end{aligned}$$

Given two paths $\mathcal{P}_1, \mathcal{P}_2$ of lengths i and j in $\mathbf{G}$ starting from o and having the same endpoint w, let $\mathcal{P}$ be the path obtained by traversing $\mathcal{P}_1$ first and returning to o via $\mathcal{P}_2$. Then

$$\mathbf{P}[\{U_k\}_{k\le i+j} = \mathcal{P}] = \mathbf{P}[\{U_k\}_{k\le i} = \mathcal{P}_1]\mathbf{P}[\{U'_k\}_{k\le j} = \mathcal{P}_2].$$

by our assumption of constant degrees. Summing over all possible w and all $\mathcal{P}_1, \mathcal{P}_2$, we get

$$\mathbf{P}[U_i = U'_j | U_0 = U'_0 = v] = \mathbf{P}[U_{i+j} = v | U_0 = v]. \tag{2}$$

Moreover, for SRW U on any infinite graph with bounded degrees,

$$\mathbf{P}[U_n = o | U_0 = o] \le \frac{C}{\sqrt{n}}, \tag{3}$$

for some constant C (not depending on o). For a proof, see Woess [8], Corollary 14.6 .

From (2) and (3) we get

$$\begin{aligned}\mathbf{P}[X_n = Y_n = (v, L) \text{ for some } v \in \mathbf{G}] &\le \mathbf{P}[V_n = V'_n = L; K_n = K'_n = 0] \\ &+ C\mathbf{E}\left[\frac{\mathbf{1}(V_n = V'_n = L; K_n + K'_n > 0)}{\sqrt{K_n + K'_n}}\right].\end{aligned} \tag{4}$$

To bound this quantity we think of V_n as being generated in the following manner. Take a simple random walk $\{S_n\}$ on $\mathbf{Z}$ (no self-loop at 0) starting from 0 and let $\{G_i\}$ be i.i.d. Geometric($\frac{d}{d+2}$) random variables. To be precise, this means that $\mathbf{P}[G_i = k] = (\frac{d}{d+2})^k \frac{2}{d+2}$ for $k \ge 0$. Then we generate V by following the path S except that at the ith visit to the origin by S, the walk V stays there for G_i steps before taking the next step according to S. Similarly, V' is generated using S' and $\{G'_i\}$.

Then let $H_n = \sum_{i=1}^{n/2} \mathbf{1}(S_i = 0)$ and similarly define H'_n. Then, either $K_n \ge R_n := \sum_{i=1}^{H_n} G_i$ or else $K_n \ge \frac{n}{2}$. Therefore the second summand on the right in (4) can be bounded by (omitting the constant C)

$$\mathbf{E}\left[\frac{\mathbf{1}(V_n = V'_n = L; R_n + R'_n > 0)}{\sqrt{(R_n \wedge \frac{n}{2}) + (R'_n \wedge \frac{n}{2})}}\right] + \mathbf{P}[V_n = V'_n = L; R_n = R'_n = 0]. \tag{5}$$

Condition on $\{S_i : i \le n/2\}$ and $\{G_i : i \le H_n\}$ and on their primed counterparts. If it happens that $\max\{R_n, R'_n\} < \frac{n}{4}$, then V and V' have at least $n/4$ more steps to go and hence the conditional probability that $V_n = V'_n = L$ is at most $\frac{C^2}{n}$ (because $\mathbf{P}[V_{n/4} = L'] \le \frac{C}{\sqrt{n}}$ for any L'). Thus the first term in (5) can be bounded by

$$C'\mathbf{E}\left[\frac{\mathbf{1}(R_n + R'_n > 0)}{\sqrt{R_n + R'_n}}\frac{1}{n} + \frac{\mathbf{1}\left(\max\{R_n, R'_n\} > \frac{n}{4}\right)}{\sqrt{n}}\right]. \tag{6}$$

We recall the following facts

- $\mathbf{E}\left[H_n^{-1/2}\mathbf{1}(H_n \ge 1)\right] \le C_1 n^{-1/4}$. To see this, consider

$$\begin{aligned}\mathbf{E}\left[H_n^{-1/2}\mathbf{1}(H_n \ge 1)\right] &\le \frac{1}{n^{1/4}} + \sum_{k=1}^{n^{1/2}} \frac{\mathbf{P}[H_n = k]}{\sqrt{k}} \\ &\le \frac{1}{n^{1/4}} + \frac{C_1}{n^{1/2}}\sum_{k=1}^{n^{1/2}}\frac{1}{\sqrt{k}} \qquad \left(\text{since } \mathbf{P}[H_n = k] \le \frac{C}{n^{1/2}} \forall k\right) \\ &\le C_1 n^{-1/4}.\end{aligned}$$

- If $\{G_i\}$ are i.i.d. Geometric(p) random variables, then

$$\mathbf{E}\left[\frac{\mathbf{1}\left(\sum_{i=1}^{r} G_i \neq 0\right)}{\sqrt{\sum_{i=1}^{r} G_i}}\right] \leq \frac{C(p)}{\sqrt{r}}. \tag{7}$$

(Let $\mu = \mathbf{E}[G_i]$. If $\frac{1}{r}\sum_{i=1}^{r} G_i > \mu - \epsilon$, then the random variable in (7) is less than $\frac{1}{\sqrt{r(\mu-\epsilon)}}$. The probability that $\frac{1}{r}\sum_{i=1}^{r} G_i$ is less than $\mu - \epsilon$ decays exponentially, by Cramér's theorem).

These facts immediately give

$$\mathbf{E}\left[\frac{\mathbf{1}(R_n + R_n' > 0)}{\sqrt{R_n + R_n'}}\right] \leq \frac{C_3}{n^{1/4}}.$$

We ultimately want to get a bound for $\mathbf{P}[X_n = Y_n = (v, L)$ for some $v \in \mathbf{G}]$. From what we have done so far this is bounded by the sum of the following three terms

- The first term in (6) is bounded by $C_4 n^{-5/4}$.
- The second term in (6) is bounded by $C_5 \mathbf{P}[R_n > \frac{n}{4}]/\sqrt{n}$. This decays super-polynomially.
- The second term in (5) and the first term in (4) are together bounded by

$$C_6 \mathbf{P}[V_n = V_n' = L, R_n = R_n' = 0].$$

To bound $\mathbf{P}[V_n = V_n' = L, R_n = R_n' = 0]$, condition on $\{S_i : i \leq \frac{n}{2}\}$, $\{G_i : i \leq \frac{n}{2}\}$ and their primed versions as before. Since the probability that a simple random walk on $\mathbf{Z}$ does not return to zero up to time n is asymptotic to $\frac{C}{\sqrt{n}}$, we can easily deduce that

$$\mathbf{P}[V_n = V_n' = L \text{ and } R_n = R_n' = 0] = O\left(\frac{1}{n^2}\right).$$

Thus we get

$$\mathbf{P}[X_n = Y_n = (v, L) \text{ for some } v \in \mathbf{G}] \leq \frac{C}{n^{5/4}} \qquad \text{for every } L \in \mathbf{Z}, n \geq 1. \tag{8}$$

Now,

$$\begin{aligned}\mathbf{E}[Z_{n,\ell}] &= \sum_{N=n}^{2n} \sum_{L=\ell}^{2\ell} \mathbf{P}[X_N = Y_N = (v, L) \text{ for some } v \in \mathbf{G}] \\ &\leq n\ell \frac{C}{n^{5/4}} = C\frac{\ell}{n^{1/4}},\end{aligned}$$

as claimed. ∎

Note that from the above lemma we also get

$$\mathbf{E}[W_{n,\ell}] \leq C\frac{\ell}{n^{1/4}} \qquad \text{for all } n, \ell \geq 1 \text{ and some constant } C < \infty. \tag{9}$$

Lemma 2.2 *Fix $0 < \alpha < 1$. There is a constant $C > 0$ (depending on α but not on ℓ or n) such that for all n, ℓ with $1 \leq \ell < 2(2n)^{1/2\alpha}$, we have $\mathbf{E}[W_{n,\ell}|A_{n,\ell}] \geq C\ell^\alpha$.*

Proof (Lemma 2.2) Suppose $A_{n,\ell}$ occurs. Then the two random walks collide at a time N with $n \leq N \leq 2n$, and at some vertex (v, L) with $\ell \leq L \leq 2\ell$. Then in $W_{n,\ell}$ we are counting all collisions that occur for the next $2n$ steps or till one of the walks reaches $(v, L \pm \frac{\ell}{2})$, whichever occurs earlier. By considering only collisions that occur before one of them hits $(v, L \pm \frac{\ell}{2})$ the problem is reduced to one about random walks on a segment of $\mathbf{Z}$.

More precisely, let U, V be two independent random walks on $\mathbf{Z}$ starting from 0. Let T_U be the first time U hits $\pm\frac{\ell}{2}$ and similarly define T_V. If

$$Y_{n,\ell} = \sum_{k=0}^{2n \wedge T_U \wedge T_V} \mathbf{1}(U_k = V_k),$$

then given that the event $A_{n,\ell}$ occurs, $W_{n,\ell}$ is stochastically larger than $Y_{n,\ell}$. Therefore, if $2n \geq (\ell/2)^{2\alpha}$, then

$$\begin{aligned}
\mathbf{E}[W_{n,\ell}|A_{n,\ell}] &\geq \mathbf{E}[Y_{n,\ell}] \\
&\geq \sum_{k=0}^{(\ell/2)^{2\alpha}} \mathbf{P}[U_k = V_k; T_U \wedge T_V > k] \\
&\geq \left(\sum_{k=0}^{(\ell/2)^{2\alpha}} \mathbf{P}[U_k = V_k] \right) - (\ell/2)^{2\alpha} \mathbf{P}[T_U \wedge T_V \leq (\ell/2)^{2\alpha}] \\
&\geq \sum_{k=0}^{(\ell/2)^{2\alpha}} \frac{C'}{\sqrt{k}} - (\ell/2)^{2\alpha} 2\mathbf{P}[T_U \leq (\ell/2)^{2\alpha}],
\end{aligned}$$

since for independent SRWs U, V on $\mathbf{Z}$, we have $\mathbf{P}[U_k = V_k] \sim C'k^{-1/2}$. Observe that $\mathbf{P}[T_U \leq (\ell/2)^{2\alpha}]$ tends to zero faster than any polynomial in ℓ. Therefore,

$$\mathbf{E}[W_{n,\ell}|A_{n,\ell}] \geq C\ell^\alpha.$$

This proves the lemma. ∎

From the two lemmas above, given $\alpha < 1$ we have constants C_1, C_2 such that

$$\mathbf{E}[W_{n,\ell}] \leq C_1 \frac{\ell}{n^{1/4}} \qquad \text{for every } \ell, n, \tag{10}$$

$$\mathbf{E}[W_{n,\ell}|A_{n,\ell}] \geq C_2 \ell^\alpha \qquad \text{for } \ell \leq 2(2n)^{1/2\alpha}, \tag{11}$$

whence we get

$$\mathbf{P}[A_{n,\ell}] \leq \frac{\mathbf{E}[W_{n,\ell}]}{\mathbf{E}[W_{n,\ell}|A_{n,\ell}]} \leq C \frac{\ell^{1-\alpha}}{n^{1/4}} \qquad \text{for } \ell \leq 2(2n)^{1/2\alpha}, \tag{12}$$

for yet another constant C.

Now we let n, ℓ satisfying $\ell \leq 2(2n)^{1/2\alpha}$ run over powers of 2, and get

$$
\begin{aligned}
\sum_{r=0}^{\infty} \sum_{k=0}^{1+\frac{r+1}{2\alpha}} \mathbf{P}[A_{2^r,2^k}] &\leq \sum_{r=0}^{\infty} \sum_{k=0}^{1+\frac{r+1}{2\alpha}} C \frac{2^{k(1-\alpha)}}{2^{r/4}} && \text{by (12)} \\
&\leq \sum_{r=0}^{\infty} C' \frac{2^{r(1-\alpha)/2\alpha}}{2^{r/4}} \\
&< \infty && \text{if } \alpha > \tfrac{2}{3}.
\end{aligned}
$$

Thus almost surely only finitely many of the events $A_{2^r,2^k}$ for $k \leq 1 + \frac{r+1}{2\alpha}$ occur. This shows that if $2/3 < \alpha < 1$, then the set

$$\{n : X_n = Y_n = (v, \ell) \text{ for some } v \in \mathbf{G} \text{ and } \ell \text{ with } 1 \leq |\ell| \leq 2(2n)^{1/2\alpha}\}$$

is finite almost surely (since each such (n, ℓ) is contained in one of the above sets). We proved this for $1 \leq \ell \leq 2(2n)^{1/2\alpha}$. By symmetry, the same holds for negative ℓ. The number of meetings on the backbone, i.e., on $\{\ell = 0\}$, is finite, by (8).

However, $\{n : |V_n| > 2(2n)^{1/2\alpha} \text{ or } |V'_n| > 2(2n)^{1/2\alpha}\}$ is finite almost surely as can be easily seen, for instance, from the law of iterated logarithm.

This proves that the total number of collisions between the two random walkers on Comb($\mathbf{G}$) is finite almost surely. ∎

3 More Examples

Definition Given two graphs $\mathbf{G}, \mathbf{H}$, and a vertex $\mathbf{v}$ of $\mathbf{H}$, define $\text{Comb}_{\mathbf{v}}(\mathbf{G}, \mathbf{H})$ to be the graph with vertex set $V(\mathbf{G}) \times V(\mathbf{H})$ and edge set

$$\{[(x, w), (x, z)] : [w, z] \text{ is an edge in } \mathbf{H}\} \cup \{[(x, \mathbf{v}), (y, \mathbf{v})] : [x, y] \text{ is an edge in } \mathbf{G}\}.$$

When $\mathbf{H} = \mathbf{Z}$ (and without loss of generality $\mathbf{v} = 0$), this clearly reduces to Comb($\mathbf{G}$).

If $\mathbf{G}, \mathbf{H}$ are recurrent, and $\mathbf{v}$ is a vertex of $\mathbf{H}$, then $\text{Comb}_{\mathbf{v}}(\mathbf{G}, \mathbf{H})$ is also obviously recurrent. When $\mathbf{H} = \mathbf{Z}^2$, we take $\mathbf{v} = (0, 0)$ and drop the subscript $\mathbf{v}$ in $\text{Comb}_{\mathbf{v}}(\mathbf{G}, \mathbf{H})$.

Theorem 3.1 *Let $\mathbf{G}$ be any recurrent infinite graph with constant vertex degree. Then Comb$(\mathbf{G}, \mathbf{Z}^2)$ has the finite collision property.*

Proof As the proof is very similar to that of Theorem 1.1 (the difference is in the estimates) we shall only briefly sketch the main steps.

For $\ell \geq 1$, let $B_\ell = \{(x, y) \in \mathbf{Z}^2 : \ell \leq \max\{|x|, |y|\} \leq 2\ell\}$ be the annulus of radii ℓ and 2ℓ. Then we define

$$Z_{n,\ell} = |\{(N, L) : n \leq N \leq 2n, L \in B_\ell \text{ and } X_N = Y_N = (v, L) \text{ for some } v \in \mathbf{G}\}|.$$

Then define $A_{n,\ell}$ and $W_{n,\ell}$ as before. Then analogously to Lemma 2.1 and Lemma 2.2 we have the following lemma.

Lemma 3.2 *With the above definitions,*

- $\mathbf{E}[W_{n,\ell}] \leq C_1 \frac{\ell^2}{n\sqrt{\log(n)}}$ $\forall 1 \leq \ell, n$.
- $\mathbf{E}[W_{n,\ell} | A_{n,\ell}] \geq C_2 \log(\ell)$ *for* $1 \leq \ell \leq n$.

Proof (Lemma 3.2) The upper bound for $\mathbf{E}[W_{n,\ell}]$ can be proved along the same lines as Lemma 2.1. First we prove

$$\mathbf{P}[X_n = Y_n = (v, L) \text{ for some } v \in \mathbf{G}] \leq \frac{C}{n^2\sqrt{\log(n)}}. \tag{13}$$

All the steps go through without change till (6). Moreover, the terms with $R_n = 0$ or $R_n \geq \frac{n}{4}$ etc can be shown to be of lower order in the same manner. (To bound the terms with $\{R_n = 0\}$, use the fact that for simple random walk in the plane, $\mathbf{P}[H_n = 0] \leq \frac{C}{\log n}$. See Erdős and Taylor [1].) Only note that in $\mathbf{Z}^2$ the n-step transition probabilities are bounded by Cn^{-1}. The dominant term is

$$\mathbf{E}\left[\frac{\mathbf{1}(V_n = V'_n = L; R_n + R'_n > 0)}{\sqrt{(R_n \wedge \frac{n}{2}) + (R'_n \wedge \frac{n}{2})}}\right],$$

where the notations are as before (now V, V' are random walks on $\mathbf{Z}^2$ instead of $\mathbf{Z}$).

For simple random walk in the plane $\mathbf{P}[H_n = k] \leq \frac{C}{\log n}$ $\forall n, k$ (H_n is the number of returns to origin by time n. See Erdős and Taylor [1]). Using this and the bound (7) for i.i.d. Geometric variables, we get the bound (13). In $W_{n,\ell}$ we are counting (up to constants) n steps and ℓ^2 sites, and thus the upper bound for $\mathbf{E}[W_{n,\ell}]$ follows.

The lower bound for $\mathbf{E}[W_{n,\ell}|A_{n,\ell}]$ is even easier. Referring back to the proof of Lemma 2.2, a lower bound can be obtained by counting only those meetings that occur for a duration of $2n$ and before one of the two walkers goes a distance of $\ell/2$ from the meeting point (that is assured by $A_{n,\ell}$). Since at least $\ell/2$ steps are needed to go a distance $\ell/2$, if $4n > \ell$,

$$\begin{aligned}
\mathbf{E}[W_{n,\ell}|A_{n,\ell}] &\geq \sum_{k=1}^{\ell/2} \mathbf{P}[U_k = V_k] \qquad U, V \text{ are SRWs on } \mathbf{Z}^2 \\
&\geq \sum_{k=1}^{\ell/2} \frac{C'}{k} \geq C_2 \log(\ell).
\end{aligned}$$

This completes the proof of Lemma 3.2. ∎

From Lemma 3.2 we get

$$\begin{aligned}
\mathbf{P}[A_{n,\ell}] &\leq \frac{\mathbf{E}[W_{n,\ell}]}{\mathbf{E}[W_{n,\ell}|A_{n,\ell}]} \\
&\leq C\frac{\ell^2}{n\sqrt{\log(n)}\log(\ell)} \qquad \text{for } 2 \leq \ell < 4n.
\end{aligned}$$

Now we let n, ℓ run over powers of 2 but only over pairs for which $2 \leq \ell \leq \sqrt{n}(\log(n))^{1/8}$ (trivially the above bound for $\mathbf{P}[A_{n,\ell}]$ holds for these values of n, ℓ). Here log denotes logarithm to base 2. Then

$$\begin{aligned}
\sum_{r=1}^{\infty} \sum_{k=1}^{\frac{r}{2}+\frac{1}{8}\log(r)} \mathbf{P}[A_{2^r,2^k}] &\leq \sum_{r=1}^{\infty} \sum_{k=1}^{\frac{r}{2}+\frac{1}{8}\log(r)} C\frac{2^{2k}}{2^r\sqrt{r}k} \\
&\leq \sum_{r=1}^{\infty} \frac{C_0}{r^{5/4}} \\
&< \infty,
\end{aligned}$$

where, in the penultimate line, we have used the following easily checked fact:

$$\sum_{k=1}^{n} \frac{4^k}{k} \leq C\frac{4^n}{n}$$

for some constant C not depending on n.

This proves that almost surely only finitely many of the events $A_{2^r,2^k}$ with $k \leq \frac{r}{2} + \frac{1}{8}\log(r)$ occur.(The cases $\ell = 0, 1$ are taken care of directly by (13).) However, as before, letting $V_n = (V_n^{(1)}, V_n^{(2)})$ and similarly for V', we observe that $\{n : \max\{|V_n^{(1)}|, |V_n^{(2)}|\} > \sqrt{n}(\log(n))^{1/8}$ or $\max\{|V_n'^{(1)}|, |V_n'^{(2)}|\} > \sqrt{n}(\log(n))^{1/8}\}$ is finite almost surely, as shown by the law of iterated logarithm. ∎

4 Questions

- Is it true for any two infinite recurrent graphs $\mathbf{G}, \mathbf{H}$ and any vertex $\mathbf{v} \in \mathbf{H}$ that $\mathrm{Comb}_{\mathbf{v}}(\mathbf{G}, \mathbf{H})$ has the finite collision property?

- If $\mathbf{H}_n$ is a sequence of finite graphs then the graph obtained by attaching $\mathbf{H}_n$ to the vertex n of $\mathbf{Z}$ gives a comb-like structure similar to the examples given in this paper. This leads us to the following questions.

  - Do trees in the uniform and minimum spanning forests on $\mathbf{Z}^d$ have the finite collision property? For definitions and properties of Uniform and Minimal Spanning forests see Lyons and Peres [6].
  - Does a critical Galton-Watson tree conditioned to survive have the same property ? (Assume that the offpring distribution has finite variance.) This conditioning on an event of zero probability can be made precise easily; see Kesten [4].

  The reason for expecting such behavior is that these trees are known to be "one-ended", meaning that they have the comb-like structure described above (although the "backbone" extends infinitely in only one direction).

- Does the incipient infinite cluster in $\mathbf{Z}^2$ (this is the cluster containing the origin in bond percolation on $\mathbf{Z}^2$ at criticality, conditioned to be infinite) have the finite collision property? It is known that almost surely there is no infinite cluster in $\mathbf{Z}^2$ at criticality. However, the incipient infinite cluster can still be defined. See Kesten [3].

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